

Magnetoelliptic Instabilities

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ABSTRACT

We consider the stability of a configuration consisting of a vertical magnetic field in a planar flow on elliptical streamlines in ideal hydromagnetics. In the absence of a magnetic field the elliptical flow is universally unstable (the “elliptical instability”). We find this universal instability persists in the presence of magnetic fields of arbitrary strength, although the growthrate decreases somewhat. We also find further instabilities due to the presence of the magnetic field. One of these, a destabilization of Alfvén waves, requires the magnetic parameter to exceed a certain critical value. A second, involving a mixing of hydrodynamic and magnetic modes, occurs for all magnetic-field strengths. These instabilities may be important in tidally distorted or otherwise elliptical disks. A disk of finite thickness is stable if the magnetic fieldstrength exceeds a critical value, similar to the fieldstrength which suppresses the magnetorotational instability.

Subject headings: accretion discs – instabilities: elliptical, hydromagnetic

1. Introduction

The problem of momentum transport in accretion disks is widely believed to require hydrodynamic or hydromagnetic turbulence for its resolution. The origin of this turbulence may be sought in the instability of laminar solutions of the equations of hydromagnetics, solutions that are compatible with the geometry of accretion disks. The recent history of these efforts has taken the form of first recognizing such an instability mechanism, and then trying to incorporate that mechanism into realistic disk models.

The magnetorotational instability (MRI) mechanism, originally discovered by Velikhov (1959) and Chandrasekhar (1960) and first applied to accretion disks in Balbus & Hawley (1991), is of this kind (see Balbus & Hawley (1998) for a review). It appears in rotating, magnetized systems in which the specific angular momentum increases outward and in which the magnetic field is weak enough that rotational effects are not overwhelmed by magnetic tension.

A second mechanism, that of the elliptical instability considered by Goodman (1993) and others (Lubow, Pringle, and Kerswell 1993; Ryu and Goodman 1994; Ryu, Goodman, and Vishniac 1996), is also consistent with the accretion-disk setting. This instability mechanism has been reviewed by Kerswell (2002). In the setting considered by Goodman et. al., it appears to require a secondary in order to enforce departure from rotational symmetry of the streamlines via a tidal potential. This is certainly appropriate for binary systems but it is likely that, even in the absence of a secondary, the laminar motion in the plane of the disk would not be accurately circular, so the elliptical-instability mechanism would appear to be a candidate of considerable generality. It does not require a magnetic field. One of the conclusions of the present paper is that it further persists in the presence of a magnetic field. In the idealized setting of the present problem, the latter may be of arbitrarily large strength. However, we also argue that in the setting of a disk geometry, there may indeed be a limit on the field strength.

In this paper we therefore investigate the interaction of a vertical magnetic field with flow on elliptical streamlines, on the ground that both magnetic fields and noncircular streamlines are likely ingredients in accretion-disk settings. There are similarities with and differences from previous work on effect of magnetic fields on the elliptical instability (Kerswell 1994), which are discussed in §6.

2. Formulation

We consider flow on elliptical streamlines together with a magnetic field and investigate linear stability theory. The underlying equations are the Euler equations of fluid dynamics

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + (\text{curl } \mathbf{B}) \times \mathbf{B} \quad (1)$$

and the induction equation

$$\mathbf{B}_t + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u}. \quad (2)$$

We shall assume that $\text{div } \mathbf{u} = 0$ and $\text{div } \mathbf{B} = 0$ and that the fluid is unbounded.

It is easy to check that the following steady fields represent a solution of the preceding system:

$$\mathbf{U} = \Omega \left(-\frac{a_1}{a_2} x_2, \frac{a_2}{a_1} x_1, 0 \right), \quad \mathbf{B} = (0, 0, B), \quad P = \frac{\Omega^2}{2} (x_1^2 + x_2^2). \quad (3)$$

Here Ω and B are constants, and a constant may also be added to the pressure term. More general exact solutions of the combined fluid/magnetic equations exist in an unbounded domain (Craik 1988); the case in hand is probably the simplest of these.

2.1. The perturbed system

Let $\mathbf{u}, \mathbf{B}, p$ be replaced by $\mathbf{U} + \mathbf{u}, \mathbf{B} + \mathbf{b}, P + p$ in equations (1, 2) above, and linearize. The resulting perturbation equations are

$$\mathbf{u}_t + \mathbf{U} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{U} = -\nabla p + (\text{curl } \mathbf{b}) \times \mathbf{B} \quad (4)$$

and

$$\mathbf{b}_t + \mathbf{U} \cdot \nabla \mathbf{b} = \mathbf{B} \cdot \nabla \mathbf{u} + \mathbf{b} \cdot \nabla \mathbf{U}, \quad (5)$$

together with the conditions that \mathbf{u} and \mathbf{b} be solenoidal. These equations allow *rotating-wave* solutions of the form

$$\mathbf{u} = \mathbf{v}(t) \exp i(\mathbf{k}(t), \mathbf{x}), \quad \mathbf{b} = \mathbf{w}(t) \exp i(\mathbf{k}(t), \mathbf{x}), \quad p = \phi(t) \exp i(\mathbf{k}(t), \mathbf{x}) \quad (6)$$

where the expression (\mathbf{k}, \mathbf{x}) denotes the inner product. Because \mathbf{u} and \mathbf{b} are solenoidal, the conditions

$$(\mathbf{k}(t), \mathbf{v}(t)) = 0, \quad (\mathbf{k}(t), \mathbf{w}(t)) = 0 \quad (7)$$

must be satisfied.

Write

$$\mathbf{U} = A\mathbf{x} \text{ where } A = \Omega \begin{pmatrix} 0 & -E & 0 \\ E^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E = a_1/a_2. \quad (8)$$

Then on substituting the rotating-wave expressions from equation (6) into the perturbation equations one finds

$$\dot{\mathbf{k}} = -A^t \mathbf{k}, \quad (9)$$

$$\dot{\mathbf{v}} = -A\mathbf{v} - i\mathbf{k}\phi + iB(\mathbf{k} \times \mathbf{w}) \times \mathbf{e}_3 \quad (10)$$

$$\dot{\mathbf{w}} = i(k_3 B)\mathbf{v} + A\mathbf{w}. \quad (11)$$

Equation (9) can be solved to give

$$\mathbf{k} = (\kappa \cos(\Omega t - \chi), E\kappa \sin(\Omega t - \chi), k_3), \quad (12)$$

where κ, k_3 and χ are constants. The pressure coefficient ϕ can be eliminated with the aid of the solenoidal condition (7). One finds

$$-i(\phi + Bw_3) = 2k^{-2}(A^t \mathbf{k}, \mathbf{v}). \quad (13)$$

The equation for \mathbf{v} now takes the form

$$\dot{\mathbf{v}} = C(t)\mathbf{v} + i(k_3 B)\mathbf{w}, \quad (14)$$

where

$$C(t) = -2(\Omega/k^2) \begin{pmatrix} -E^{-1}k_1k_2 & Ek_1^2 & 0 \\ -E^{-1}k_2^2 & Ek_1k_2 & 0 \\ -E^{-1}k_3k_2 & Ek_1k_3 & 0 \end{pmatrix} + \Omega \begin{pmatrix} 0 & E & 0 \\ -E^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (15)$$

It's convenient to break the six-dimensional system consisting of equations (14) and (11) into two, one of size four and the other of size two:

$$\begin{aligned} \dot{v}_1 &= -(2\Omega/k^2)k_1(Ek_1v_2 - E^{-1}k_2v_1) + \Omega Ev_2 + imw_1, \\ \dot{v}_2 &= -(2\Omega/k^2)k_2(Ek_1v_2 - E^{-1}k_2v_1) - \Omega E^{-1}v_1 + imw_2, \\ \dot{w}_1 &= imv_1 - \Omega Ew_2, \\ \dot{w}_2 &= imv_2 + \Omega E^{-1}w_1, \end{aligned}$$

where $m = k_3 B$. These four equations are self-contained and the remaining equations,

$$\dot{v}_3 = -(2\Omega/k^2)k_3(Ek_1v_2 - E^{-1}k_2v_1) + imw_3, \quad (16)$$

$$\dot{w}_3 = imv_3 \quad (17)$$

may be integrated once the expression

$$c_1 \equiv Ek_1v_2 - E^{-1}k_2v_1 \quad (18)$$

is found by solving the four-dimensional system above. Equations (9), (10), (11) and (13) imply that

$$\frac{d}{dt}(\mathbf{k}, \mathbf{v}) = im(\mathbf{k}, \mathbf{w}) \text{ and } \frac{d}{dt}(\mathbf{k}, \mathbf{w}) = im(\mathbf{k}, \mathbf{v}).$$

Thus in solving this system we need to impose the conditions that these inner products are zero initially; this will thereafter maintain the incompressibility conditions (7).

The incompressibility condition provides an alternative way of finding v_3 and w_3 once the equations for v_1, v_2, w_1, w_2 have been solved, provided that $k_3 \neq 0$. The only cases for which k_3 can vanish are those for which the combinations $k_1v_1 + k_2v_2$ and $k_1w_1 + k_2w_2$ are also found to vanish on solving the four-dimensional system above. It is not difficult to show that there can be no instability associated with such a solution (see in particular the equivalent system (20) below). Accordingly, we henceforth consider only perturbations with vertical wave number $k_3 \neq 0$.

2.2. Change of variables

We change to new variables to facilitate subsequent calculations¹.

$$\begin{aligned} c_1 &= Ek_1v_2 - E^{-1}k_2v_1, \\ c_2 &= k_1v_1 + k_2v_2 (= -k_3v_3), \\ c_3 &= Ek_1w_2 - E^{-1}k_2w_1, \\ c_4 &= k_1w_1 + k_2w_2 (= -k_3w_3). \end{aligned} \quad (19)$$

This is a time-dependent (periodic) change of variables since \mathbf{k} is periodic in t . The equations to be solved take the form

$$\dot{c} = D(t) c \quad (20)$$

in these variables, with (we have put $\Omega = 1$ here to agree with earlier conventions)

$$D(t) = \begin{pmatrix} -2(E - E^{-1})k^{-2}k_1k_2 & -2 & im & 0 \\ 2k^{-2}k_3^2 & 0 & 0 & im \\ im & 0 & 0 & 0 \\ 0 & im & 0 & 0 \end{pmatrix}. \quad (21)$$

¹The origin of this change of variables is related to the existence of the rotating-wave solutions.

2.3. General Considerations

The coefficients of the matrix D depend on the phase angle χ appearing in the expressions (12) above for the wave vector k . For purposes of studying stability, we may set $\chi = 0^2$. This is easily seen by making the substitution $t' = t - \chi$, which eliminates χ from the equation. For the remainder of this work we take $\chi = 0$.

The system (20) presents a Floquet problem (cf. Yakubovich and Starzhinskii (1975)): the stability of the chosen steady solution \mathbf{U}, \mathbf{B} depends on whether there are solutions of this system that grow exponentially with time. This is settled by finding the Floquet multiplier matrix M . The latter is defined as follows. Let $\Phi(t)$ be the fundamental matrix solution of equation (20) that reduces to the identity at $t = 0$. Then, since the periodicity of D is 2π , $M = \Phi(2\pi)$. If any eigenvalue λ of M has modulus exceeding one, this implies that there is indeed an exponentially growing solution.

It is familiar in conservative problems that

Proposition 1 *Whenever λ is an eigenvalue of the Floquet matrix, so also are its inverse λ^{-1} and its complex conjugate $\bar{\lambda}$.*

The first statement of this proposition is a typically a consequence of canonical Hamiltonian structure, the second a consequence of the reality of the underlying problem. However, the system (20) is not canonical, and the matrix appearing in it is not real. We can nevertheless establish these familiar properties of the eigenvalues directly from the system (20), as follows. The time-reversal invariance of the physical problem is reflected in the existence of a reversing symmetry $R = \text{diag}(1, -1, -1, 1)$ of the matrix D above: $RD(-t) = -D(t)R$, implying that whenever $c(t)$ is a solution so also is $Rc(-t)$. Since the solutions of the Floquet problem have the structure $c(t) = p(t) \exp(\sigma t)$, there must also be a solution $p(-t) \exp(-\sigma t)$. But the eigenvalues of the Floquet multiplier matrix M are the values $\lambda = \exp(2\pi\sigma)$. This shows that if λ is an eigenvalue of M , so also is $\exp(-2\pi\sigma) = \lambda^{-1}$.

Similarly, under matrix transformation $S = \text{diag}(1, 1, -1, -1)$, D goes to its complex conjugate. This shows that $M = S\bar{M}S^{-1}$, i.e., that M and its conjugate have the same eigenvalues.

Immediate consequences of Proposition 1 are the following: first, in the stable case, eigenvalues of M lie on the unit circle; second, if, as parameters change, an eigenvalue is

²On the other hand, for purposes of solving the initial-value problem, which involves integrating over initial wave vectors, we would need to retain it.

at the onset of instability, it must have multiplicity two (or higher). The latter conclusion is because, in the complex λ -plane, the dangerous eigenvalue λ must leave the unit circle simultaneously with $\bar{\lambda}^{-1}$, which lies along the same ray as λ and therefore coincides with it when they both lie on the unit circle. Thus a necessary condition for the onset of linear instability is a resonance where two Floquet multipliers coincide.

2.4. Parameters

There are three dimensionless parameters that figure in this problem. We call them ϵ , μ , and η :

$$\epsilon = \frac{1}{2} (E - E^{-1}), \quad \mu = k_3/k_0, \quad \text{and} \quad \eta = k_0 B. \quad (22)$$

Thus ϵ represents the departure of the streamlines of the unperturbed flow from axial symmetry. In these equations $k_0 = \sqrt{\kappa^2 + k_3^2}$ and represents the length of the wave vector if $\epsilon = 0$. The magnetic parameter η depends not only on the strength of the unperturbed magnetic field but also on the wavelength of the perturbation.

In the matrix D above, the magnetic field enters through the parameter $m = k_3 B$ (which is a measure of the magnetic tension force), and we shall continue to use this notation for the present, on the understanding that $m = \mu\eta$. It is also clear that we can use E rather than ϵ to measure the departure from rotational symmetry, and we shall do this in some cases.

3. Analysis

The Floquet matrix is

$$M(\epsilon, \mu, \eta) = \Phi(2\pi, \epsilon, \mu, \eta).$$

One could map out the stability and instability regions in the $\epsilon\mu\eta$ parameter space numerically by integrating the system (20) systematically for many values of these parameters. We in fact do this for a selection of parameter values in §4 below. However, in this and the following section we present the outlines and results of an asymptotic analysis based on regarding ϵ as a small parameter (details are presented in the Appendices). This is more revealing than the numerical results on their own. It is also of considerable importance in interpreting the numerical results, and is quite accurate even for values of ϵ that are not very small (cf. Figure 1 below). The calculation proceeds in two steps; finding M , and calculating its eigenvalues.

3.1. The Floquet Matrix

The asymptotic analysis is facilitated by the circumstance that, if we put $\epsilon = 0$, the coefficient matrix D_0 (say) of equation (21) becomes constant. We find the eigenvalues and eigenvectors of D_0 in Appendix A.1. There are two complex conjugate pairs of modes. One pair reduces to the ordinary hydrodynamic modes in the limit $\eta \rightarrow 0$. The second pair are magnetic modes with zero frequency at $\eta = 0$. In the weak field limit, the ratio of magnetic to kinetic energy is $\mathcal{O}(\eta^2/4)$ for the modified hydrodynamic modes and $\mathcal{O}(4/\eta^2)$ for the magnetic modes. If $\eta \gg 1$, the kinetic and magnetic energies are near equipartition for both types of mode.

We now turn to a brief description of the asymptotic (or perturbation) procedure. In what follows the parameter η will be held fixed so, to simplify the notation, we suppress the dependence of the Floquet matrix on this parameter: $M = M(\epsilon, \mu) = \Phi(2\pi, \epsilon, \mu)$. We shall need the Taylor expansion

$$M(\epsilon, \mu) = M(0, \mu_0) + M_\epsilon(0, \mu_0)\epsilon + M_\mu(0, \mu_0)(\mu - \mu_0) + \cdots, \quad (23)$$

where the dots indicate higher-order terms in ϵ and $\mu - \mu_0$. The reason for allowing variations with μ as well as variations with ϵ is that the region in the $\epsilon\mu$ plane where instability occurs is typically a wedge with apex at a point $(\epsilon, \mu) = (0, \mu_0)$ and boundaries $\mu = \mu_0 + \nu_\pm\epsilon$, where the slopes ν_+ and ν_- are to be found. (cf. Figure 1 below). We will therefore consider μ of the form

$$\mu = \mu_0 + \nu\epsilon + \cdots \quad (24)$$

where ν may be regarded as a fixed parameter to be chosen later. This will lead us to the widest of wedges in the $\epsilon\mu$ plane, of width $O(\epsilon)$, excluding other wedges of width $O(\epsilon^m)$ with $m \geq 2$. These higher-order wedges typically occupy a tiny fraction of the parameter space (see Figure 4 below). As a result of the representation (24) we may write

$$M = M_0 + M_1\epsilon + \cdots \quad (25)$$

where

$$M_0 = M(0, \mu_0) \text{ and } M_1 = M_\epsilon(0, \mu_0) + \nu M_\mu(0, \mu_0). \quad (26)$$

The matrix M_0 is given in Appendix A.1, together with an expression for M_ϵ . We effect a change of basis, which diagonalizes M_0 and simplifies the calculation of M_ϵ , in Appendix A.2. The elements of M_1 are calculated in Appendix A.4.

3.2. The characteristic polynomial

Denote by

$$p(\lambda, \epsilon) = |M(\epsilon) - \lambda I| \quad (27)$$

the characteristic polynomial of the Floquet multiplier matrix $M(\epsilon)$ and its roots by $\Lambda_1, \Lambda_2, \dots$. A necessary condition for stability is that each root lie on the unit circle. To explore the onset of instability for various parameter values, we shall obtain the characteristic polynomial p in the form

$$p(\lambda, \epsilon) = p_0(\lambda) + p_1(\lambda)\epsilon + p_2(\lambda)\epsilon^2 + \dots \quad (28)$$

and exploit our knowledge of the roots of p_0 to obtain the roots of $p(\lambda, \epsilon)$ in the form of a Puiseux expansion in ϵ (Hille 1972)).

The nature of this expansion depends on the multiplicities of the roots $\{\lambda_k\}$ of p_0 , the characteristic polynomial of the unperturbed Floquet matrix M_0 . These roots are given by the expressions $\lambda_k = \exp 2\pi\sigma_k$ where the $\{\sigma_k\}$ are the eigenvalues of the matrix D_0 given in Appendix A.1 (equation A3); they are all distinct. However, it is possible for the multipliers $\{\lambda_k\}$ to be repeated even when, as in the present case, the $\{\sigma_k\}$ are distinct: if $\sigma_k - \sigma_l = ik$ for an integer $k \neq 0$, then $\lambda_k = \lambda_l$.

A necessary condition for the onset of instability is that there be a double (or higher) root of the characteristic equation, and we henceforth restrict consideration to the case of double roots³. For definiteness, we suppose $\lambda_2 = \lambda_1$. Then the Puiseux expansion takes the form

$$\Lambda_1(\epsilon) = \lambda_1 + \epsilon^{1/2}\beta_{1/2} + \epsilon\beta_1 + \dots \quad (29)$$

Substituting into the characteristic equation (and taking into account that p_0 and p'_0 both vanish at λ_1) yields for the coefficient $\beta_{1/2}$ of the leading-order correction the equation

$$\beta_{1/2}^2 = -2p_1(\lambda_1)/p''_0(\lambda_1). \quad (30)$$

The two values of $\beta_{1/2}$ give the generic expressions for the change in a double eigenvalue, yielding a pair of roots branching from the double root λ_1 . However, if $p_1(\lambda_1) = 0$, this expression is inadequate and one must proceed to the next term in order to determine the effect of the perturbation on the stability. Under the present assumptions, it is indeed the case that p_1 vanishes at λ_1 , as we show in Appendix A.3⁴.

³Higher order zeros are not ruled out in this problem, since there are three independent parameters, but we do not pursue this here.

⁴This “nongeneric” behavior can be traced to the circumstance that the perturbation expansion takes place at a codimension-two point, i.e., where two relations must hold among the parameters.

We must suppose then that the expansion of $p(\lambda, \epsilon)$ is carried out to second order in ϵ :

$$p(\lambda, \epsilon) = p_0(\lambda) + \epsilon p_1(\lambda) + \epsilon^2 p_2(\lambda) + \dots \quad (31)$$

Then in the Puiseux expansion above $\beta_{1/2} = 0$ so $\Lambda_1 = \lambda_1 + \beta_1 \epsilon + \dots$, and β_1 is found by solving the quadratic equation

$$\frac{1}{2} p_0''(\lambda_1) \beta_1^2 + p_1'(\lambda_1) \beta_1 + p_2(\lambda_1) = 0. \quad (32)$$

In the case at hand, the common value of λ_1 and λ_2 lies on the unit circle. In order for the perturbed values of the Floquet multipliers to lie *off* the unit circle (and therefore imply instability), it is easy to verify that it is necessary and sufficient that β_1/λ_1 have a nonvanishing real part. Thus if we define

$$\alpha = \beta_1/\lambda_1, \quad (33)$$

we have the following criterion:

Proposition 2 *Either α is pure-imaginary and we infer stability (to leading order in ϵ), or $\text{Re } \alpha \neq 0$ and we infer instability.*

The magnitude of the real part of α is also related to the growthrate of the instability. If we define an instability increment

$$\Delta = |\Lambda| - 1, \quad (34)$$

then $\Delta = \epsilon \text{Re } \alpha$ and the growthrate is equal to $\Delta/2\pi$, to leading order in ϵ .

The long calculations that lead to the coefficients appearing in equation (32) are carried out in the Appendices. In the notation employed there, equation (32) therefore takes the form

$$\alpha^2 - \left\{ \tilde{J}_{11} + \tilde{J}_{22} + \frac{2\pi i}{\mu} \nu [\omega_1 + \omega_2] \right\} \alpha + \left| \begin{array}{cc} \tilde{J}_{11} + 2\pi\nu\sigma_1/\mu & \tilde{J}_{12} \\ \tilde{J}_{21} & \tilde{J}_{22} + 2\pi\nu\sigma_2/\mu \end{array} \right| = 0, \quad (35)$$

In equation (35), α has the meaning of Proposition 2 above, and the symbols \tilde{J}_{ij} are defined in Appendix A.2 (equation A15). There are obvious modifications of this formula if $\lambda_k = \lambda_l$ instead of $\lambda_1 = \lambda_2$.

3.3. The Resonant Cases

The resonant cases for $\epsilon = 0$ (circular streamlines) are those parameter values (μ, η) such that $\omega_j - \omega_l = k$, where k is an integer. We'll find that these can be written in the form $\mu = f(\eta)$ (e.g., equation (37) below). Since $\epsilon = 0$, the μ values in question are those that were designated μ_0 in equation (24) above. We no longer need the designation μ_0 and, in the relations below, use the symbol μ in its place.

If $k \neq \pm 2$ the matrix \tilde{J} is diagonal (see equation A25 below) and equation (35) has the roots $\alpha = J_{11} + 2\pi i \nu \omega_1 / \mu$ and $\alpha = J_{22} + 2\pi i \nu \omega_2 / \mu$ where $\sigma_j = i\omega_j$ for $j = 1, 2, 3, 4$. It is easy to check that these diagonal entries are pure-imaginary (cf. equations A26 and A9 below) and therefore, in accordance with Proposition 2, there is no instability to leading order in ϵ . We therefore now consider the only cases ($k = \pm 2$) that can lead to instability to this order.

Recall that the original parameters of the problem are ϵ – representing the departure from axial symmetry of the undisturbed streamlines – $\mu = k_3/k_0$ – representing the vertical wavenumber – and $\eta = k_0 B$. The auxiliary parameters $m = \mu\eta$ and $q = \mu\sqrt{1 + \eta^2}$ are introduced to simplify the notation. With the frequencies taken in the order

$$(\omega_1, \omega_2, \omega_3, \omega_4) = (\mu + q, -\mu - q, \mu - q, -\mu + q) \quad (36)$$

the replacement $\mu \rightarrow -\mu$ results in the same frequencies in the opposite order. We may therefore assume without loss of generality that $\mu > 0$. Further scrutiny of the formula (36) shows that we need only consider the following four, distinct, $k = 2$ resonances.

3.3.1. Case 1. $\omega_1 - \omega_2 = 2$

The resonant modes are those that reduce, when $\eta = 0$, to the purely hydrodynamic modes. This case therefore represents the modification, due to the presence of the vertical magnetic field, of the universal elliptical instability. In this case $\omega_1 = -\omega_2 = \mu + q = 1$, implying that

$$\mu = \frac{1}{1 + \sqrt{1 + \eta^2}}. \quad (37)$$

This ratio changes from $1/2$ at $\eta = 0$ to 0 as $\eta \rightarrow \infty$. Evaluating the integrals defining (\tilde{J}_{ij}) (equation A15 below) and solving equation (35) above, we find

$$\alpha^2 = \left[\frac{\pi}{2} (1 + \mu)^2 \right]^2 - \pi^2 \left[\frac{2\nu}{\mu} - \mu (1 + \mu) \right]^2. \quad (38)$$

This has a maximum instability increment (when $\nu = \mu^2(1 + \mu)/2$) given (suppressing a factor of ϵ) by

$$\alpha_{max} = \frac{\pi}{2} (1 + \mu)^2.$$

In the pure-hydrodynamic limit $\eta = 0$ we find $\mu = 1/2$ and therefore $\alpha_{max} = 9\pi/8$. Since the growthrate is given by $\alpha\epsilon/2\pi$, this gives a maximum growthrate of $9\epsilon/16$, in agreement with previous results obtained by other methods (Waleffe 1990; Goodman 1993). In the limit $\eta \rightarrow \infty$ this maximum instability increment tends to the finite limit $\pi/2$, about half its value in the pure-hydrodynamic limit.

This instability has a *bandwidth* $(\nu_+ - \nu_-)\epsilon$ which is, for given ϵ and η , the length of the μ -interval for which the unperturbed configuration is unstable. It is determined by the values of ν that make the real part of α vanish. These may be read off equation (38) above:

$$\nu_+ = \mu(1 + \mu)(1 + 3\mu)/4, \quad \nu_- = -\mu(1 - \mu^2)/4.$$

In the limit $\eta \rightarrow 0$ (the pure-hydrodynamic case) these give $\nu_+ = 15/32$ and $\nu_- = -3/32$. These values of ν_{\pm} can also be inferred from Waleffe's treatment of the pure-hydrodynamic case. In the limit of large magnetic parameter η , the width of the band tends to zero.

3.3.2. Case 2. $\omega_1 - \omega_3 = 2$

The resonant modes consist of a hydrodynamic mode and a purely magnetic mode (one frequency would vanish if $\eta = 0$). In this case $q = 1$, implying that $\omega_1 = \mu + 1$, $\omega_3 = \mu - 1$ and

$$\mu = \frac{1}{\sqrt{1 + \eta^2}}. \quad (39)$$

Thus the ratio μ changes from 1 at $\eta = 0$ to 0 as $\eta \rightarrow \infty$. This represents a 'mixed mode', i.e., the resonance is between a purely hydrodynamic mode and a purely magnetic one. Evaluating the integrals and solving the quadratic (35), we find

$$\alpha = i\pi\nu \left(\frac{2\nu}{\mu} - 1 + \mu^2 \right) \pm \sqrt{D} \quad (40)$$

where

$$D = -\pi^2 \left[\frac{2\nu}{\mu} - \frac{1}{2}(1 - \mu^4) \right] \left[\frac{2\nu}{\mu} - \frac{1}{2}(1 - \mu^2)(3\mu^2 - 1) \right].$$

If $D < 0$ then α is pure-imaginary and the unperturbed configuration is stable, so instability prevails if and only if $D > 0$.

Instability can indeed occur and has its maximal increment when $\nu = \mu^3(1 - \mu^2)/2$. This maximal increment is (except for a factor of ϵ)

$$(\text{Re } \alpha)_{max} = \frac{\pi}{2} (1 - \mu)^2.$$

This ranges from 0 when $\eta = 0$ to $\pi/2$ as $\eta \rightarrow \infty$.

The width of instability band can be calculated as in the preceding case by finding the values of μ for which $D = 0$. These are

$$\nu_+ = \frac{1}{4}\mu(1 - \mu^4) \text{ and } \nu_- = \frac{1}{4}\mu(1 - \mu^2)(3\mu^2 - 1).$$

3.3.3. Case 3. $\omega_4 - \omega_3 = 2$

These are the purely magnetic modes, which play no role if $\eta = 0$. For this case we have $q - \mu = 1$ implying, for fixed η that

$$\mu = \frac{1}{\sqrt{1 + \eta^2 - 1}}. \quad (41)$$

Since μ cannot exceed 1, this resonance can only occur for sufficiently large values of η , namely

$$\eta > \sqrt{3}. \quad (42)$$

For a given wavelength k_0 this would require a sufficiently large magnetic field B . When this condition is satisfied, we have $\omega_4 = 1 = -\omega_3$. The formula for the instability increment α becomes

$$\alpha^2 = \left[\frac{\pi}{2} (1 - \mu)^2 \right]^2 - \pi^2 \left[\frac{2\nu}{\mu} + \mu(1 - \mu) \right]^2. \quad (43)$$

The maximum instability increment for given ϵ is (again suppressing a factor of ϵ)

$$\alpha_{max} = \frac{\pi}{2} (1 - \mu)^2,$$

which occurs for $\nu = -\mu^2(1 - \mu)/2$. It vanishes in the limit $\eta = 0$ and tends to $\pi/2$ as $\eta \rightarrow \infty$. The upper and lower edges of the band of instability are expressed by the formulas $\mu = \mu_0 + \nu_{\pm}\epsilon$ where ν_{\pm} may be determined from equation (43) by requiring α to vanish. One finds

$$\nu_+ = \frac{1}{4}\mu(1 - \mu)(1 - 3\mu) \text{ and } \nu_- = -\frac{1}{4}\mu(1 - \mu^2).$$

The bandwidth is therefore

$$\nu_+ - \nu_- = \frac{1}{2}\mu(1 - \mu)^2.$$

This vanishes both in the limit as $\eta \rightarrow 0$ and in the limit as $\eta \rightarrow \infty$. The maximum bandwidth occurs when $\mu = 1/3$ or $\eta = \sqrt{15}$.

3.3.4. Case 4. $\omega_1 - \omega_4 = 2$

For this $\mu = 1$. It is clear from the expressions for the matrix J that the latter vanishes in this case. This leads to pure-imaginary values of α and therefore there is no instability associated with this resonance.

4. Numerical results

In this section we present a selection of numerical results. These are obtained by integrating the system (20) to obtain the fundamental matrix solution $\Phi(t, \epsilon, \mu, \eta)$ and evaluating it at $t = 2\pi$ to get the Floquet matrix $M(\epsilon, \mu, \eta)$. For fixed η , the eigenvalue of maximum modulus is found as a function of E (rather than ϵ) and μ , and the regions of the $E\mu$ plane where this maximum modulus exceeds one are distinguished. We have carried this out to $E = 1.6$ ($\epsilon = 0.4875$) in the figures although there is no limitation on the size of E , or of ϵ , in this method.

In Figure 1, we have taken the magnetic parameter equal to zero in the left-hand panel, so this represents the purely hydrodynamic case studied originally by Bayly (1986) and Pierrehumbert (1986) and subsequently by others. For $\eta > 0$, a mixed mode of interaction involving both hydrodynamic and hydromagnetic modes comes into existence, and this is shown in the right-hand panel of Figure 1, where $\eta = 1$. This is too small for the remaining leading-order instability to appear. That remaining instability, which represents a resonant interaction between two modes that owe their existence to the presence of the magnetic field, is indicated in Figure 2 for a magnetic parameter $\eta = 2$ (right-hand panel), slightly greater than the minimum value ($\sqrt{3}$) for the existence of this instability.

The asymptotic formulas imply that the maximal growthrate (or equivalently the maximal instability increment Δ) for each of the wedges of instability tends to a fixed value as the magnetic parameter η increases. This is illustrated in Figure 3. However, the asymptotic formulas for the growthrates are less accurate than those for the stability boundaries, for the larger values of E .

In identifying these tongues, we have made repeated use of the asymptotic formulas presented earlier. The numerical procedure also picks up some further tongues, related to higher-order resonances, that are excluded by the procedure leading to the asymptotic formulas. These we have mostly ignored on the ground that they are too weak and occupy too small a region of the parameter space to be significant, but we show one such resonance tongue for two values of η in Figure 4

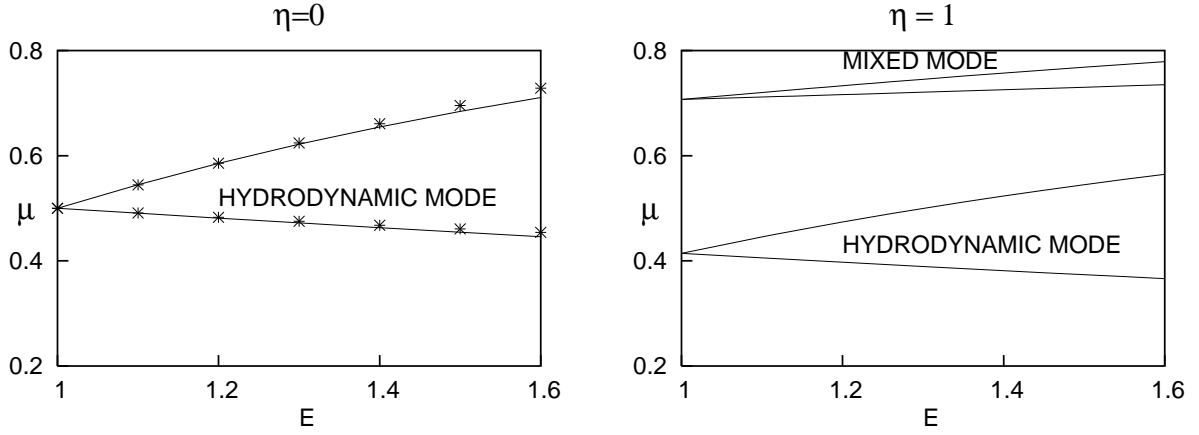


Fig. 1.— On the left is the case $\eta = 0$, i.e., the purely hydrodynamic case considered earlier by several other authors. It is shown here to contrast it with the case when the magnetic field is not zero. The lines of asterisks indicate the same stability boundaries obtained from the asymptotic formulas $\mu = \mu_0 + \nu_{\pm}\epsilon$; this approximation is seen to be quite good. It is typical of the other cases considered. On the right is the case $\eta = 1$, showing both the effect of the magnetic field on the hydrodynamic mode and the existence of a new mode of instability that is due to the presence of the magnetic field.

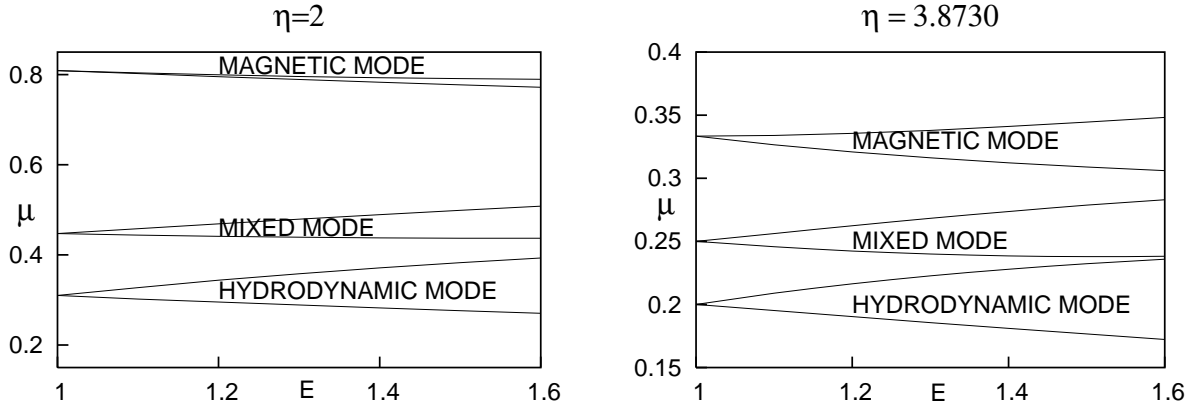


Fig. 2.— On the left is the case $\eta = 2$. Three regions of instability occur here. The lowest of these is the modification of the hydrodynamic-mode instability indicated in Figure 1. The middle region refers to the mixed hydrodynamic-magnetic mode. The uppermost region, very thin and labeled MAGNETIC MODE, exists only for values of η exceeding $\sqrt{3}$, so is poorly developed for this value of η . On the right is the case $\eta = \sqrt{15} \approx 3.873$, for which the width of the uppermost, purely magnetic, mode band is at its greatest. Note however that the vertical scale is compressed relative to the diagram on the left, exaggerating the widths of these bands by about a factor of two.

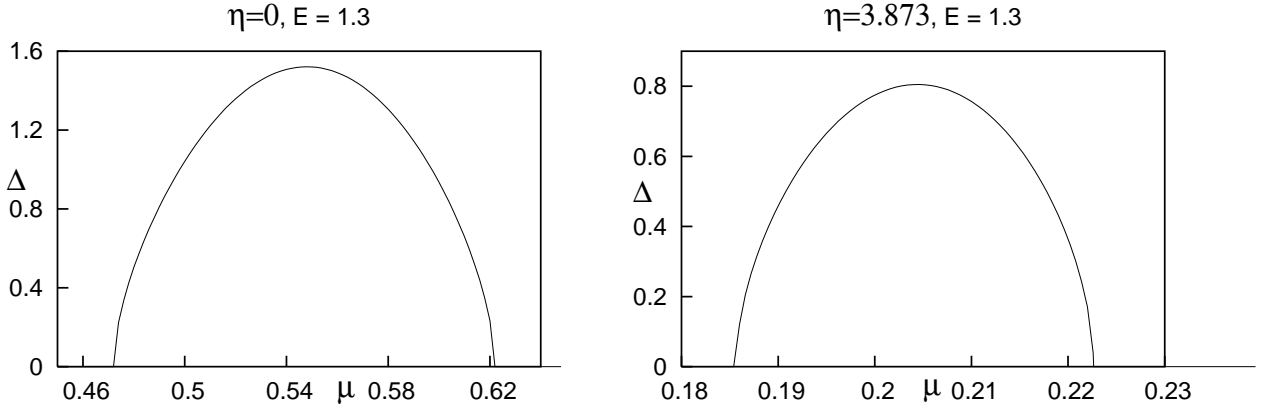


Fig. 3.— The left-hand panel shows the instability increment Δ as a function of μ for a fixed value of the magnetic parameter ($\eta = 0$) and the ellipticity ($E = 1.3$). The right-hand panel does the same except that $\eta = \sqrt{15}$.

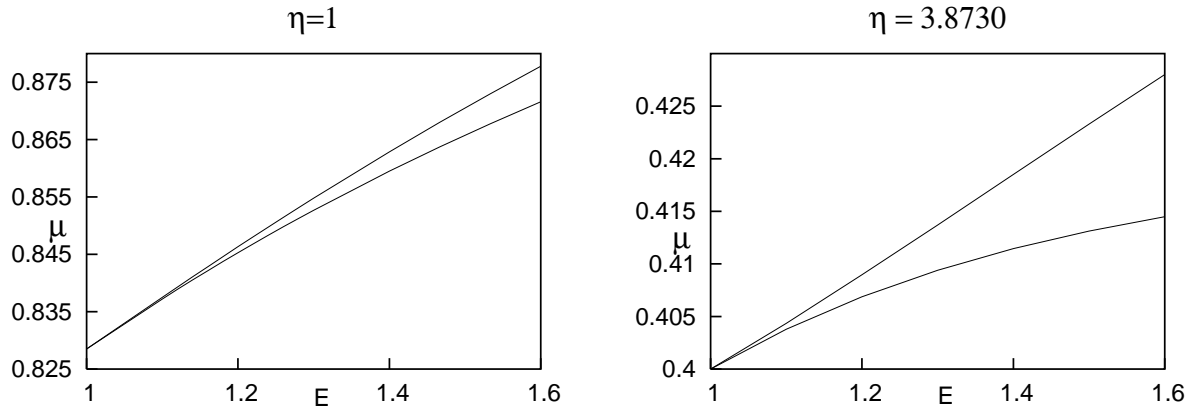


Fig. 4.— These show higher order wedges of instability. Both are cases of resonance between the two hydrodynamic modes but with $\omega_1 - \omega_2 = 4$, rather than 2. The left-hand panel shows the instability wedge when $\eta = 1$, the right-hand panel shows the corresponding wedge when $\eta = \sqrt{15}$.

5. Upper bound on fieldstrength

The results of §§3.3 and 4 show that magnetoelliptic instability occurs in vertically unbounded systems whatever the fieldstrength, and that the growth rates are relatively insensitive to the magnetic fieldstrength parameter η . The loci of instability in the (E, μ) plane, however, do depend on η . As $\eta \rightarrow \infty$, the resonant μ , in all three cases, scales as η^{-1} and the magnetic tension parameter $m \rightarrow 1$. This is a consequence of the resonant character of the instability. Since the mode frequency must be close to the rotational frequency, the Alfvén frequency m cannot be too large. As $B \rightarrow \infty$, k_3 must approach zero.

In a system of finite vertical thickness H , k_3 cannot drop below χ/H , where χ is a factor of order unity. Therefore, if v_A exceeds $\Omega H/\chi$, we expect the magnetoelliptic instability to be suppressed. A precise upper bound on v_A follows from eqn. (A3) and application of the resonance conditions; equations (37), (39), and (41). In Cases 1 and 2, instability requires $m^2 \leq 1$. Case 3 requires $m^2 \leq 3$. Therefore, magnetoelliptic instability requires $v_A \leq \sqrt{3}\Omega H/\chi$ (there is a correction of order ϵ due to the finite bandwidth). The corresponding fieldstrength is comparable to the maximum value of the field at which the magnetorotational instability can operate (Balbus & Hawley 1998).

6. Discussion

We have explored the effect of a uniform, vertical magnetic field on the stability of planar, incompressible flow with elliptical streamlines in an unbounded medium, in the approximation of ideal magnetohydrodynamics. In the absence of magnetic fields, flows with elliptical streamlines having ellipticity parameter ϵ [see equation (22)] are known to be unstable to perturbations with wavevectors that are transverse to the plane of the flow (the “elliptical instability”). Our first conclusion is that this elliptical instability persists in the presence of a vertical magnetic field: the latter decreases the maximum growthrate but fails to suppress the instability, no matter how large the magnetic-field parameter becomes. It can be compared with the conclusion of Kerswell (1994) that a toroidal magnetic field has a stabilizing influence. Kerswell’s analysis holds for small ϵ only and shows that the growthrate decreases with magnetic field in that limit. Our result, which holds for a vertical magnetic field, shows a similar trend for small values of the magnetic-field parameter, but this trend never results in complete stabilization of the elliptical instability with increasing magnetic field.

A second conclusion is that there are further instabilities associated with the presence of the magnetic field. One of these, for which the eigenvector is a mixture of hydrodynamic

and magnetic modes, occurs for all values of the magnetic field parameter. Another, for which the eigenvector is a combination of magnetic modes only, sets in for values of the magnetic-field parameter exceeding a certain threshold value ($\eta > \sqrt{3}$). In all three of these instabilities, for large magnetic fields, the wave vector makes only a small angle with the plane of the unperturbed flow, reflecting the familiar tendency for dynamics to become nearly 2-dimensional in a strong, well ordered magnetic field. This is reflected in Figure 2, which shows that as η increases, the unstable wedges are pushed to smaller μ . Although the unstable fraction of the (E, μ) plane decreases with increasing η (except for a very slight maximum at $\eta \sim 2.18$, reflecting the onset of instability between magnetic modes), the separation between the unstable wedges also decreases. While the nonlinear evolution of the unstable system is beyond the scope of this work, the destabilization of a nearly continuous swath of parameter space may have consequences for the interactions between unstable modes. In all three cases, the maximum instability increment tends to $\epsilon\pi/2$, i.e., the maximum growthrate of the unstable modes tends, in dimensional units, to $\epsilon\Omega/4$.

All three of these instabilities may be relevant in accretion-disk settings. In systems of finite thickness H , however, the instability is suppressed if the Alfvén speed v_A exceeds a critical value of order ΩH . Magnetorotational instabilities are quenched at approximately the same fieldstrength (Balbus & Hawley 1998). The growth rate of magnetoelliptical instabilities is smaller than that of magnetorotational instabilities by a factor of order ϵ , and thus they are not necessarily the primary instability in magnetized disks. They may well play a secondary role by breaking up eddies or vortices generated by other mechanisms.

Magnetoelliptic instabilities may also occur in the inner parts of barred galaxies, in which the gas flow is slightly elliptical and the magnetic field, at least in the Milky Way, has a vertical component (Morris and Serabyn 1996). In such settings, the instabilities could be a source of turbulence, possibly affecting the mass supply to a central compact object.

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Appendices

We carry out calculations leading to the coefficients appearing in equation (32) in a series of steps. Since $p(\lambda) = |M - \lambda I|$, with M given by equations (25) and (26), we begin with the expression for M .

A. The Expansion for M

A.1. Zero-order Problem and Solution for M

The matrix $D(t, \epsilon, \mu)$ has only two entries that depend on ϵ : D_{11} and D_{21} . If, therefore, we write the Taylor expansion

$$D(t, \epsilon, \mu) = D_0(t, \mu) + \epsilon D_\epsilon(t, \mu) + \dots \quad (\text{A1})$$

then

$$D_0(t, \mu) = D(t, 0, \mu) \text{ and } D_\epsilon(t, \mu) = D_\epsilon(t, 0, \mu),$$

where $D_\epsilon = \partial D / \partial \epsilon$. For D_0 we find the constant matrix

$$D_0(t, \mu) = D_0(\mu) = \begin{pmatrix} 0 & -2 & im & 0 \\ 2\mu^2 & 0 & 0 & im \\ im & 0 & 0 & 0 \\ 0 & im & 0 & 0 \end{pmatrix}. \quad (\text{A2})$$

Its eigenvalues are

$$\sigma_1 = i(\mu + q), \sigma_2 = -i(\mu + q), \sigma_3 = i(\mu - q), \sigma_4 = -i(\mu - q), \quad (\text{A3})$$

where $q = \sqrt{\mu^2 + m^2} = \mu\sqrt{1 + \eta^2}$. These are distinct and nonzero as long as $\mu \neq 0$ and $\eta \neq 0$, which we assume to be the case. The first two correspond to “hydrodynamic modes” since they reduce, when $\eta = 0$, to the eigenvalues of the purely hydrodynamic case. The second two refer to “magnetic modes” since they are zero in that limit. These are all of stable type, corresponding to frequencies ω_k , $k = 1, \dots, 4$. Regarding the matrix D_ϵ , one can easily work it out from the expression (21) above: all its entries vanish except $(D_\epsilon)_{11}$ and $(D_\epsilon)_{21}$. One finds

$$(D_\epsilon)_{11} = i(1 - \mu^2)(e^{2it} - e^{-2it}), \quad (\text{A4})$$

and

$$(D_\epsilon)_{21} = \mu^2(1 - \mu^2)(e^{2it} + e^{-2it} - 2) + 2\mu\nu. \quad (\text{A5})$$

From the matrices D_0 and D_ϵ we can construct the matrices $M_0(\mu)$ and $M_\epsilon(\mu)$ needed in the formula (26) for M_1 . For M_0 we simply have $\exp 2\pi D_0$. For M_ϵ we proceed as follows. On the finite time-interval $[0, 2\pi]$ we may write

$$\Phi(t, \epsilon, \mu) = \Phi_0(t, \mu) + \epsilon \Phi_1(t, \mu) + \dots, \quad \Phi_1(0, \mu) = 0.$$

Substituting this in the differential equation (20), expanding to first order in ϵ , using the variation of constants formula (cf. Coddington and Levinson (1955)) and setting $t = 2\pi$, we get

$$M(\epsilon, \mu) = M_0(\mu) \left(I + \epsilon \int_0^{2\pi} \Phi_0^{-1}(s, \mu) D_\epsilon(s, \mu) \Phi_0(s, \mu) ds \right). \quad (\text{A6})$$

This expresses the Floquet matrix correctly to linear order in ϵ , and this will turn out to be sufficient for our purpose. The formula above identifies $M_\epsilon(0, \mu)$:

$$M_\epsilon(0, \mu) = M_0(\mu) \int_0^T \Phi_0^{-1}(s, \mu) D_\epsilon(s, \mu) \Phi_0(s, \mu) ds. \quad (\text{A7})$$

We next proceed to simplify this expression.

A.2. A further transformation

The characteristic polynomial given in equation (27) above is the same in any coordinate system, so we shall choose one to simplify the unperturbed Floquet matrix $M_0(\mu)$.

If $\mu \neq 0$ and $\eta \neq 0$ the eigenvalues $\{\sigma_k\}$ given by equation (A3) are all distinct, so the eigenvectors are linearly independent and the matrix $T(\mu)$ formed from their columns diagonalizes D_0 :

$$\tilde{D}_0 = \text{diag}(\sigma_1, \sigma_2, \sigma_3, \sigma_4),$$

where the tilde indicates the transformed matrix: $\tilde{D} = T^{-1}DT$. We shall need to know T and T^{-1} explicitly. It is a straightforward matter to show that any eigenvector of D_0 must have the structure (up to a constant multiple)

$$\xi = \begin{pmatrix} \sigma \\ -(\sigma^2 + m^2)/2 \\ im \\ -im(\sigma^2 + m^2)/2\sigma \end{pmatrix}.$$

Substituting the particular values of σ given in equation (A3) gives the four columns of the matrix T , and from this we can construct its inverse. One finds

$$T = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ -i\mu\sigma_1 & i\mu\sigma_2 & -i\mu\sigma_3 & i\mu\sigma_4 \\ im & im & im & im \\ m\mu & -m\mu & m\mu & -m\mu \end{pmatrix}, \quad (\text{A8})$$

and

$$T^{-1} = \frac{1}{4mq} \begin{pmatrix} -im & m/\mu & \sigma_3 & i\sigma_3/\mu \\ im & m/\mu & \sigma_3 & -i\sigma_3/\mu \\ im & -m/\mu & -\sigma_1 & -i\sigma_1/\mu \\ -im & -m/\mu & -\sigma_1 & i\sigma_1/\mu \end{pmatrix}. \quad (\text{A9})$$

The matrices T and T^{-1} depend on μ and on η through the parameters

$$m = \mu\eta \text{ and } q = \sqrt{\mu^2 + m^2} = \mu\sqrt{1 + \eta^2}. \quad (\text{A10})$$

In place of equation (A6) we now obtain

$$\tilde{M} = \tilde{M}_0 \left(I + \epsilon \tilde{J} \right), \quad (\text{A11})$$

where

$$\tilde{J}(\mu) = \int_0^{2\pi} \tilde{\Phi}^{-1}(t, \mu) \tilde{D}_1(t, \mu) \tilde{\Phi}(t, \mu) dt. \quad (\text{A12})$$

Because the eigenvalues $\{\sigma_k\}$ are distinct, the matrix $\tilde{\Phi} = \exp\{\tilde{D}_0 t\}$ takes the simple, diagonal form

$$\tilde{\Phi}(t) = \text{diag}(\exp(\sigma_1 t), \exp(\sigma_2 t), \exp(\sigma_3 t), \exp(\sigma_4 t)) \quad (\text{A13})$$

The ij entry of the matrix \tilde{D}_1 is (since D_ϵ has only two nonzero entries)

$$\left(\tilde{D}_1\right)_{ij} = T_{1j} \left((T^{-1})_{i1} (D_\epsilon)_{11} + (T^{-1})_{i2} (D_\epsilon)_{21} \right). \quad (\text{A14})$$

As a result, the ij entry of the matrix \tilde{J} providing the leading-order perturbation of the Floquet matrix is

$$\begin{aligned} \tilde{J}_{ij} &= T_{1j} (T^{-1})_{i1} \int_0^{2\pi} e^{(\sigma_j - \sigma_i)t} (D_\epsilon)_{11}(t) dt \\ &+ T_{1j} (T^{-1})_{i2} \int_0^{2\pi} e^{(\sigma_j - \sigma_i)t} (D_\epsilon)_{21}(t) dt. \end{aligned} \quad (\text{A15})$$

This enables us to find $\tilde{M}_\epsilon(0, \mu)$.

For the matrix $\tilde{M}_0(\mu)$ we have the expression

$$\tilde{M}_0(\mu) = \tilde{M}(0, \mu) = \text{diag}(\lambda_1(\mu), \dots, \lambda_4(\mu)) \quad (\text{A16})$$

with $\lambda_k = 2\pi\sigma_k$. Recall that in order to construct \tilde{M}_1 we need also the derivative of this matrix with respect to μ ,

$$\tilde{M}'_0(\mu) = \tilde{M}'_\mu(0, \mu) = \text{diag}(\lambda'_1(\mu), \dots, \lambda'_4(\mu)). \quad (\text{A17})$$

According to equations (A3) and (A10), each eigenvalue σ_k of D_0 is linear in μ . Therefore $\sigma'_k(\mu) = \sigma_k(\mu)/\mu$. Since $\lambda_k = \exp(2\pi\sigma_k)$, we have

$$\lambda'_k(\mu) = \lambda_k(\mu) 2\pi\sigma_k(\mu)/\mu = \lambda_k 2\pi i \omega_k / \mu. \quad (\text{A18})$$

The formulas of this section allow one to determine the matrices \tilde{M}_0 and \tilde{M}_1 . To produce from these the coefficients $p_j(\lambda)$ appearing in equations (31) and (32) above, we need formulas for the derivatives of a determinant. These are presented in Section B below and applied in the following section.

A.3. The Expansion for $p(\lambda, \epsilon)$

We can now find the required expansion for $p(\lambda, \epsilon)$ by identifying the matrix A of Section B below with $\tilde{M} - \lambda I$ and the coefficients q_k with the coefficients $p_k(\lambda)$ of equation (31). We obtain these coefficients by writing $a_k = \lambda_k - \lambda$, $A'_{kl}(0) = \left(\tilde{M}_1\right)_{kl}$ and $A''_{kl}(0) = 2\left(\tilde{M}_2\right)_{kl}$. This gives for p_1 the following expression (with $n = 4$):

$$\begin{aligned} p_1(\lambda) = & \left(\tilde{M}_1\right)_{11} (\lambda_2 - \lambda) (\lambda_3 - \lambda) (\lambda_4 - \lambda) \\ & + \left(\tilde{M}_1\right)_{22} (\lambda_1 - \lambda) (\lambda_3 - \lambda) (\lambda_4 - \lambda) + \\ & + \left(\tilde{M}_1\right)_{33} (\lambda_1 - \lambda) (\lambda_2 - \lambda) (\lambda_4 - \lambda) \\ & + \left(\tilde{M}_1\right)_{44} (\lambda_1 - \lambda) (\lambda_2 - \lambda) (\lambda_3 - \lambda), \end{aligned} \quad (\text{A19})$$

and there is a similar, lengthier expression for p_2 obtained by making the corresponding substitutions in equation (B5) below.

The development thus far has required no assumptions regarding the multipliers $\{\lambda_k\}$. We now suppose that $\lambda_1 = \lambda_2$. It is then clear from the expression above that $p_1(\lambda_1) = 0$, as asserted in Section 3.2. The coefficients appearing in equation (32) are now easily found to be

$$p_0''(\lambda_1) = 2(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1), \quad (\text{A20})$$

$$p_1'(\lambda_1) = -\left\{\left(\tilde{M}_1\right)_{11} + \left(\tilde{M}_1\right)_{22}\right\}(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1) \quad (\text{A21})$$

and

$$p_2(\lambda_1) = \left| \begin{pmatrix} \tilde{M}_1 \\ \tilde{M}_1 \end{pmatrix}_{11} \quad \begin{pmatrix} \tilde{M}_1 \\ \tilde{M}_1 \end{pmatrix}_{12} \right| (\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1). \quad (\text{A22})$$

Equation (32) therefore takes the form

$$\beta_1^2 + \left\{\left(\tilde{M}_1\right)_{11} - \left(\tilde{M}_1\right)_{22}\right\}\beta_1 + \left| \begin{pmatrix} \tilde{M}_1 \\ \tilde{M}_1 \end{pmatrix}_{11} \quad \begin{pmatrix} \tilde{M}_1 \\ \tilde{M}_1 \end{pmatrix}_{12} \right| = 0. \quad (\text{A23})$$

We note that the calculation of the perturbation of λ_1 to first order in ϵ , which requires expanding p to second order, requires the expansion of the Floquet matrix $\tilde{M}(\epsilon) = \tilde{M}_0 + \tilde{M}_1\epsilon + \dots$ only to first order.

We have assumed that the coincident roots are the first two, $\lambda_1 = \lambda_2$. If instead we should have $\lambda_k = \lambda_l$, equation (A23) is modified by the replacement $(1, 2) \rightarrow (k, l)$.

Equation (A23), together with equations (A11), (A16), (A17) and (A18) leads to equation (35) of the text. What remains is to evaluate the integrals defining \tilde{J} , and we now turn to this.

A.4. Calculating the Elements of \tilde{M}_1

By equations (26) and (A7) above (see also equation A12), the matrix \tilde{M}_1 is given by the formula

$$\tilde{M}_1 = \tilde{M}_0(\mu) \tilde{J} + \nu \tilde{M}_\mu(0, \mu) \quad (\text{A24})$$

where the entries of \tilde{J} are given by equation (A15). From equations (A4) and (A5) it is a straightforward matter to carry out the integrations. We'll use for T the matrix given above in equation (A8). Since for this matrix $T_{1j} = \sigma_j = i\omega_j$, the formula for the entries of \tilde{J} becomes

$$\tilde{J}_{ij} = \sigma_j \left\{ (T^{-1})_{i1} \int_0^{2\pi} e^{(\sigma_j - \sigma_i)t} (D_\epsilon)_{11}(t) dt + (T^{-1})_{i2} \int_0^{2\pi} e^{(\sigma_j - \sigma_i)t} (D_\epsilon)_{21}(t) dt \right\}. \quad (\text{A25})$$

For the diagonal entries the exponential factors in the integrand reduce to unity and one finds

$$\tilde{J}_{jj} = -4\pi\mu^2 (1 - \mu^2) \sigma_j (T^{-1})_{j2}, \quad j = 1, 2, 3, 4. \quad (\text{A26})$$

For the off-diagonal entries, the formulas may be found generally, but we need them only in the resonant cases where, for some pair of indices (i, j) , $\sigma_i - \sigma_j = ki$ for a non-zero integer k .⁵ It is clear from the formulas (A4) (A5) and (A25) that only resonances with $k = \pm 2$ contribute off-diagonal terms to leading order in ϵ since for any other choice of k the integrals vanish: for $k \neq 2$ the matrix \tilde{J} is diagonal.

B. Determinantal Derivatives

Consider an $n \times n$ matrix $A(\epsilon)$ having the properties that it is a smooth function of ϵ and is diagonal at $\epsilon = 0$: $A(0) = \text{diag}(a_1, a_2, \dots, a_n)$.

We need the coefficients in the Taylor expansion of $\det A(\epsilon) \equiv q(\epsilon)$:

$$q(\epsilon) = q_0 + q_1\epsilon + q_2\epsilon^2 + \dots \quad (\text{B1})$$

where

$$q_0 = |A(0)|, \quad q_1 = \frac{d|A(\epsilon)|}{d\epsilon} \Big|_{\epsilon=0}, \quad q_2 = \frac{1}{2} \frac{d^2|A(\epsilon)|}{d\epsilon^2} \Big|_{\epsilon=0}, \quad \dots \quad (\text{B2})$$

Straightforward applications of the formula for the derivative of a determinant show that

$$q_1 = A'_{11}a_2a_3 \cdots a_n + A'_{22}a_1a_3 \cdots a_n + \cdots + A'_{nn}a_1a_2 \cdots a_{n-1}, \quad (\text{B3})$$

$$\begin{aligned} q_2 = & \frac{1}{2} [A''_{11}a_2a_3 \cdots a_n + A''_{22}a_1a_3 \cdots a_n + \cdots + A''_{nn}a_1a_2 \cdots a_{n-1}] \\ & + \begin{vmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{vmatrix} a_3a_4 \cdots a_n + \begin{vmatrix} A'_{11} & A'_{13} \\ A'_{31} & A'_{33} \end{vmatrix} a_2a_4 \cdots a_n \end{aligned} \quad (\text{B4})$$

$$+ \cdots + \begin{vmatrix} A'_{(n-1)(n-1)} & A'_{(n-1)n} \\ A'_{n(n-1)} & A'_{nn} \end{vmatrix} a_1a_2 \cdots a_{n-2}, \quad (\text{B5})$$

where the terms involving two-by-two determinants represent the sum over $k < l$ of the product of the $\{a_j\}$, a_k and a_l omitted, with the determinant

$$\begin{vmatrix} A'_{kk} & A'_{kl} \\ A'_{lk} & A'_{ll} \end{vmatrix}$$

and all derivatives are evaluated at the origin.

⁵Recall that $\sigma_i \neq \sigma_j$ for any pair i, j so $k = 0$ is excluded.

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